A Comment on the Chaotic Behaviour of van der Pol Equations with an External Periodic Excitation

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The limit cycle system with an external periodic force $\mathrm{d}^2u/\mathrm{d}t^2 - a(1-u^2)\,\mathrm{d}u/\mathrm{d}t + u^n = k\cos(\Omega t)$ ($n = 1, 3, 5, \ldots$) can show chaotic behaviour for certain values of a, k and Ω. We study the influence of n on the chaotic behaviour. For $n = 1$ we select values which result in chaotic motion of the system. Then we investigate the behaviour of the system for $n = 3$, 5 and 7. Introducing the nonlinearity u^n ($n = 3, 5, 7$) gives the surprising result that the chaotic motion ceases to exist.

It is well known that the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + f(u)\frac{\mathrm{d}u}{\mathrm{d}t} + g(u) = 0 \tag{1}$$

shows (stable) limit cycle behaviour if the functions f and g satisfy the following conditions [1]:

i) f is even; g is odd, both are continuous for all u, and $f(0) < 0$;
ii) $u\,g(u) > 0$ for $u \neq 0$;
iii) g is Lipschitzian;
iv) $F(u) \to \pm\infty$ as $u \to \pm\infty$, where

$$F(u) = \int_0^u f(s)\,\mathrm{d}s;$$

v) F has a single positive zero at $u = a$ and is monotone increasing for $u \geq a$. The equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - a(1-u^2)\frac{\mathrm{d}u}{\mathrm{d}t} + u^n = 0 \tag{2}$$

with n odd and $a > 0$ satisfies these conditions; i.e., we find stable limit cycle behaviour. For $n = 1$ we obtain the famous van der Pol equation [2]. The cases with $n = 3, 5, \ldots$ are sometimes called generalized van der Pol equations.

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In the present note we study (2) with an external periodic force, i.e.,

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - a(1-u^2)\frac{\mathrm{d}u}{\mathrm{d}t} + u^n = k\cos(\Omega t). \tag{3}$$

We are in particular interested in the chaotic behaviour in dependence on n and a. The chaotic behaviour is characterized by the maximal one dimensional Lyapunov exponent. To find the maximal one-dimensional Lyapunov exponent, system (3) together with its variational equation is integrated.

Parlitz and Lauterborn [3] found chaotic behaviour for the case $n = 1$. For example, $a = 5$, $k = 5$, and $\Omega = 2.466$ lead to chaotic motion. For $n = 3$ and $n = 5$ chaotic behaviour has been found by Ueda and Akamatsu [4] and Steeb and Kunick [5]. For example, if $n = 3$ one finds chaotic motion for $a = 0.2$, $k = 17$, and $\Omega = 4$. The chaotic bahaviour of the van der Pol equation with an external periodic force is restricted to large a, for example $a = 5$. For small a no chaotic motion exists. This is obvious since for small values of a the stable limit cycle of the van der Pol equation is approximately given by $u(t) = 2\sin t$, i.e., the limit cycle is nearly a circle. On the other hand, for $n = 3$ chaotic motion appears for small values of a and appropriate values of k and Ω. Then the nonlinear term u^3 is responsible for the chaotic behaviour. It is known that the equation $\ddot{u} + u^3 = k\cos(\Omega t)$ can show chaotic motion.

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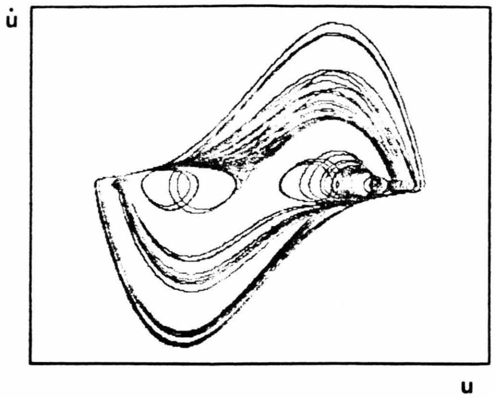

Fig. 1. Phase portrait of Van der Pol equation with external periodic force where $a = 5$, $k = 5$, $\Omega = 2.466$, and $n = 1$ (100 cycles).

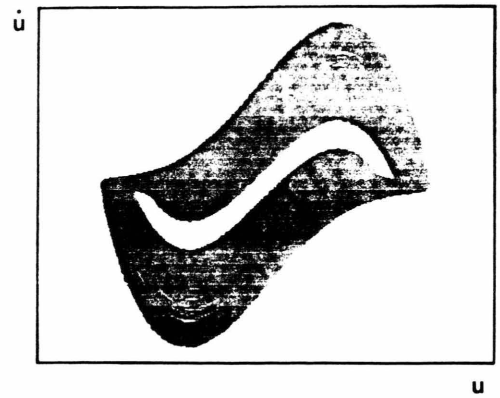

Fig. 2. Phase portrait of Van der Pol equation with external periodic force where $a = 5$, $k = 5$, $\Omega = 2.466$, and $n = 1$ (3000 cycles).

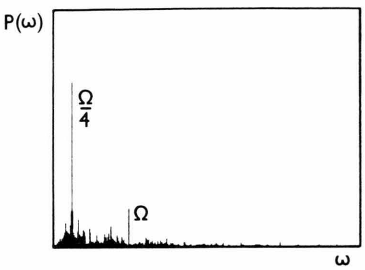

Fig. 3. Power spectrum of Van der Pol equation with external periodic force where $a = 5$, $k = 5$, $\Omega = 2.466$, and $n = 1$.

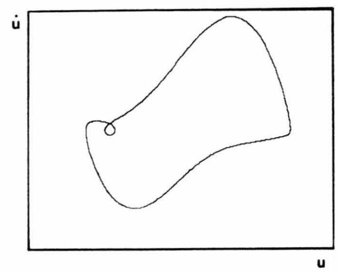

Fig. 4. Phase portrait of generalized Van der Pol equation with external periodic force where $a = 5$, $k = 5$, $\Omega = 2.466$, and $n = 3$.

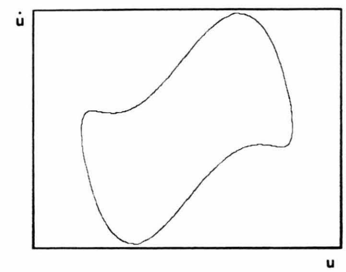

Fig. 5. Phase portrait of generalized Van der Pol equation with external periodic force where $a = 5$, $k = 5$, $\Omega = 2.466$, and $n = 5$.

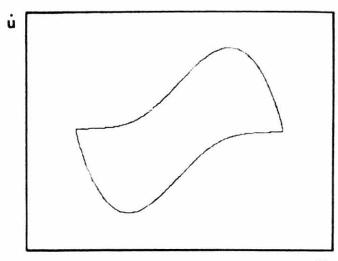

Fig. 6. Phase portrait of Van der Pol equation where $a = 5$, and $n = 1$.

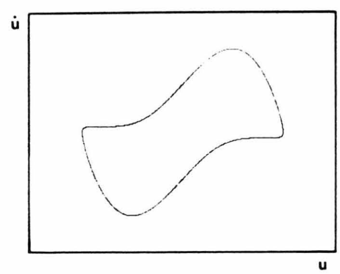

Fig. 7. Phase portrait of generalized Van der Pol equation where $a = 5$, and $n = 3$.

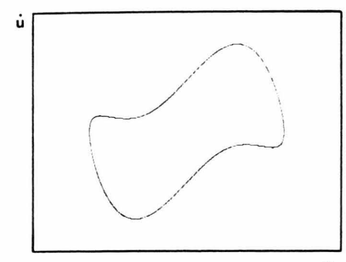

Fig. 8. Phase portrait of generalized Van der Pol equation where $a = 5$, and $n = 5$.

Figure 1 shows the phase portrait of (3) with $a = 5$, $k = 5$, $\Omega = 2.466$ and $n = 1$. The maximal one-dimensional Lyapunov exponent is positive and thus chaotic behaviour is indicated. Figure 2 shows the phase portrait for a very long computer simulation. In Fig. 3 the power spectrum is depicted. The power spectrum involves the subharmonic $\Omega/4$, together with the background "noise" (period 4-chaos).

Let us now investigate $a = 5$, $k = 5$, and $\Omega = 2.466$ for $n = 3, 5, 7$. One would expect that "introducing" an additional nonlinearity would strengthen the chaotic behaviour for these values. However, we find that the chaotic motion disappears for $n = 3$, $n = 5$ and $n = 7$ and retains the periodic motion. Figure 4 gives the phase portrait for $n = 3$ and Fig. 5 for $n = 5$. We have also calculated the power spectrum, maximal one-dimensional Lyapunov exponent and Poincaré section. No chaotic motion is indicated. For $n = 3$ the power spectrum involves the subharmonic $\Omega/2$ and the frequencies $3\Omega/2$, $5\Omega/2$, $7\Omega/2$, etc. with decreasing amplitude. Furthermore, one finds the higher harmonics 2Ω, 3Ω, 4Ω, etc. with decreasing amplitude. The subharmonic $\Omega/2$ is related to the small loop in the limit cycle (see Figure 4). For $n = 5$ no subharmonics exist and we only find higher harmonics, namely 3Ω, 5Ω, etc. The case $n = 7$ is qualitatively the same as $n = 5$.

This behaviour can be explained as follows. For large a (say $a = 5$) the phase portrait of the van der Pol equation shows two "sharp corners" at $u = 2$ and $u = -2$ (Figure 6). These two "sharp corners", generated by the "damping nonlinearity", are responsible for the chaotic behaviour when we introduce the external periodic force with appropriate values of k and Ω. The "sharp corners" disappear (the corners become smoother and smoother) when $n = 3, 5, 7$ for $a = 5$. Figure 7 shows the phase portrait of (2) with $n = 5$ and $a = 5$ (Figure 8). This smoothing effect of the nonlinearity u^n destroys the chaotic motion. The nonlinearity u^n with $n = 3, 5, 7$ can be derived from the potential $V(u) \propto u^{n+1}$. With increasing n the "particle" becomes more and more confined.

The behaviour of the disappearing chaotic motion also arises for other values of Ω in the range $2.424 \leq \Omega \leq 2.502$, with chaotic behaviour for $n = 1$.

The "sharp corners" also disappear when the value of a becomes small. The stable limit cycle becomes nearly a circle for small values of a. For small a no chaotic behaviour can be found for the van der Pol equation with an external periodic force [4, 5]. In this case the nonlinearity $u^n (n = 3, 5, 7)$ must come into play to find chaotic motion.

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